Three-dimensional stellarator codes

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Three-dimensional computer codes have been used to develop quasisymmetric stellarators with modular coils that are promising candidates for a magnetic fusion reactor. The mathematics of plasma confinement raises serious questions about the numerical calculations. Convergence studies have been performed to assess the best configurations. Comparisons with recent data from large stellarator experiments serve to validate the theory.

Modular stellarators can be viewed as an advanced tokamak hybrid appropriate for implementation as a fusion reactor (1). Quasisymmetry of the magnetic spectrum is predicted to give good confinement at high temperatures, and adequate rotational transform from the external magnetic field is expected to stabilize the plasma. New configurations have been designed by making imaginative use of three-dimensional computer codes. Because the mathematics of these stellarators is complicated, we have performed convergence studies applicable to proof of principle experiments that are being planned.

We are primarily concerned with the NSTAB equilibrium and stability code and the TRAN Monte Carlo transport code developed at New York University by Octavio Betancourt and Mark Taylor (2–5). The codes are applied to a compact stellarator called the MHH2 that has two field periods and excellent quasiaxial symmetry. For the calculations we selected a configuration with realistic physical parameters that provide good convergence, enabling us to perform long runs and make estimates of numerical errors. Theoretical conclusions can be drawn that are relevant to a wider range of examples, such as the optimized stellarator specified in Table 1.

The NSTAB code is a computer implementation of the variational principle of ideal magnetohydrodynamics (MHD). If $\mathbf{B}$ is the magnetic field and p is the pressure, solutions of the magnetostatics equations are found by minimizing the potential energy

$$E = \int\int\int [B^2/2 - p]dV$$

in a coordinate system compatible with toroidal geometry in three dimensions. An accurate finite difference scheme is used in the radial direction, and dependence on the poloidal and toroidal angles is handled by the spectral method. It is assumed that there are nested toroidal flux surfaces, and the differential equations are written in a conservation form that captures islands and current sheets. The resolution is so good that questions of stability can be settled by a mountain pass theorem asserting that when more than one solution of the problem can be found then an unstable equilibrium must exist corresponding to a saddle point in the energy landscape. Bifurcated equilibria are calculated whose magnetic surfaces have Poincaré sections displaying the structure of the most unstable modes.

The TRAN code uses a split time algorithm to calculate the confinement time of test particles by alternately tracking guiding center orbits and applying a random walk that represents collisions. The magnetic field $\mathbf{B}$ and the flow field $\mathbf{U}$ of the plasma in a background obtained by using NSTAB are held fixed during iterations that impose quasineutrality. Conservation of momentum might be enforced by the selection of $\mathbf{U}$, but because it is not thermal $\mathbf{U}$ has little effect on the collision operator. However, anomalous transport can be modeled by iterating on small variations of the electric potential within the magnetic surfaces to achieve quasineutrality between the distributions of ions and electrons. The method simulates complicated transport in stellarators remarkably well, and numerical results have been obtained for the large helical device (LHD) experiment at the National Institute for Fusion Studies in Toki, Japan that are in excellent agreement with recent observations of the energy confinement time at high temperatures (6, 7). A vectorized version of the TRAN code runs efficiently on standard work stations.

Convergence Studies

Calculation of toroidal equilibrium of a plasma without two-dimensional symmetry is a problem in mathematics that is not well posed. In terms of the toroidal flux s and a pair of angular flux coordinates θ and ϕ, the parallel current has a Fourier expansion of the form

$$\frac{\mathbf{J}\cdot\mathbf{B}}{B^2} = p' \sum \frac{m - In}{n - \iota m} B_{mn} \cos\left[\frac{(m - In)\theta + (n - \iota m)\phi}{1 - \iota I}\right],$$

where B_{mn} is the magnetic spectrum of Fourier coefficients of $1/B^2$, ι is the rotational transform, and I is the net current (8). A configuration is called quasisymmetric if a single row, column, or diagonal of the double array B_{mn} dominates that spectrum.

The small denominators $n - \iota m$ vanish at resonant surfaces where ι is rational, so smooth solutions of the partial differential equations describing MHD equilibrium do not in general exist in three dimensions. In numerical work this leads us to construct weak, discontinuous solutions of discrete equations that are expressed in conservation form, but even the best methods only converge in an asymptotic sense. Enough spectral terms must be included to eliminate significant truncation error, but not so many that the results become meaningless. Our convergence studies for the NSTAB code clarify how this can be accomplished in practice.

For convergence studies we have selected an example of the MHH2 compact stellarator, shown in Fig. 1, that has two field periods, plasma aspect ratio three, excellent quasiaxial symmetry, and a limit near 5% for the average value of $\beta = 2p/B^2$. The separatrix has a smooth shape that facilitates making long runs of the NSTAB code. The coordinate system for the computations is chosen to rotate once in the poloidal direction over a full circuit of the device in the toroidal direction. Zoning of the poloidal angle has been adjusted to provide good resolution on crude grids and reasonable spacing of the mesh both at the separatrix and the magnetic axis, which is determined by a robust algorithm. However, numerical errors appear at the magnetic axis if the radial mesh is too fine.

Applying an accelerated method of steepest descent to the variational principle, we test for stability by examining runs of the NSTAB code in which some dangerous mode has been triggered by introducing temporarily an appropriate forcing term. A run predicts stability if the mode decays during further

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Abbreviations: LHD, large helical device; MHD, magnetohydrodynamics; W7-AS, Wendelstein 7-AS.

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Table 1. Fourier coefficients Δ_{mn} of an MHH2 stellarator defined in cylindrical and toroidal coordinates by the formula $r + iz = e^{iu} \sum \Delta_{mn}\, e^{-imu+2inv}$

$m \backslash n$	−1	0	1	2	3
−1	0.190	0.130	−0.015	0.000	0.000
0	0.000	1.000	0.000	0.000	0.000
1	0.150	3.000	0.250	0.050	0.000
2	0.000	−0.090	−0.420	−0.070	0.000
3	0.000	0.000	−0.040	0.080	0.000
4	0.000	0.015	0.000	−0.015	−0.015

iterations, but if it grows then the equilibrium is unstable. A more convincing conclusion can be drawn from the mountain pass theorem if the iterations converge to a bifurcated solution whose stellarator symmetry is visibly broken by magnetic surfaces that exhibit the structure of the dangerous mode. The numerical results depend on the maximum degree N of factors in each of two angular coordinates that specify the spectral terms included in the computation. The purpose of our convergence study is to decide whether the prediction about stability approaches a meaningful limit as N increases.

For the example of the MHH2 we have performed equilibrium and stability runs of the NSTAB code with between 14 and 28 mesh intervals in the radial flux coordinate s and with a largest degree N of the spectral terms ranging as high as 48. Because of the nested surface hypothesis, islands of small width are captured better on crude grids, and the calculations are relatively insensitive to the radial mesh size because the finite difference scheme in s has an especially accurate conservation form. A preconditioned iterative scheme in the code was tuned to emphasize steepest descent of the energy over speed in decay of the residuals so that the test of stability would become more reliable. Runs were continued as long as possible to achieve accuracy sufficient for a convergence study. Recent advances in computer technology have enabled us to do this economically.

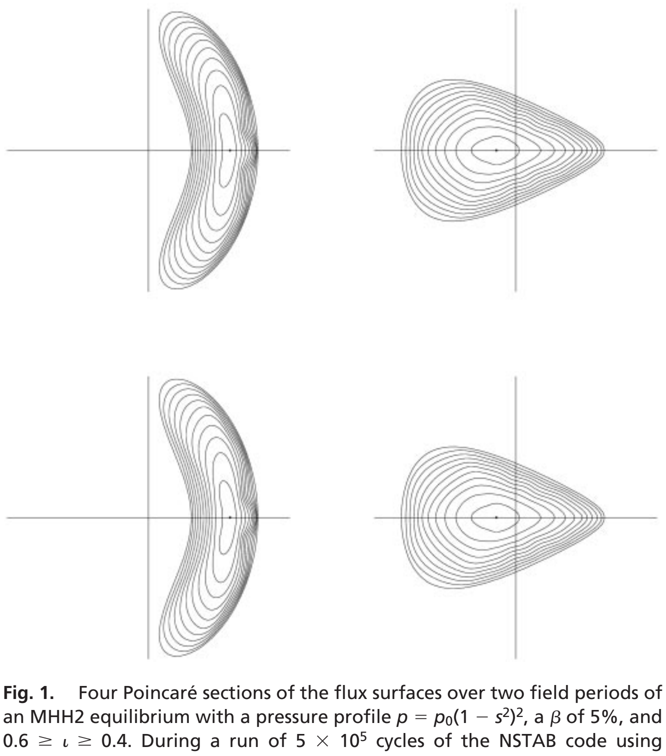

Fig. 1. Four Poincaré sections of the flux surfaces over two field periods of an MHH2 equilibrium with a pressure profile $p = p_0(1 - s^2)^2$, a β of 5%, and $0.6 \geq \iota \geq 0.4$. During a run of 5×10^5 cycles of the NSTAB code using trigonometric polynomials of degree 20, an asymmetric ballooning mode converged to symmetric ripples over regions of bad curvature in a wall stabilized solution of the magnetostatic equations that minimizes the potential energy E.

Table 2. Relationship between the highest degree N of the spectral terms used in a run of NSTAB and the resulting critical value of β that is calculated

N	16	20	24	32	48
β	0.060	0.050	0.045	0.040	0.039

In Table 2 we compare the degree N of the spectral calculations with the corresponding estimate of an average β limit based on the mountain pass theorem. If the degree is too low the method does not provide meaningful results about stability. However, for $n = 24$ one obtains efficiently answers that are of sufficient accuracy to optimize the design of a quasisymmetric stellarator like the MHH2. The results do not change significantly for $N > 32$, which is a good value to choose from the point of view of asymptotic convergence. At $N = 48$ a stage is reached where the numerical method may fail in long runs because of the singular behavior of the solution.

Table 3 shows how the convergence of the iterative scheme used in the NSTAB code depends on the degree N of the spectral terms that are used. For crude grids it is easy to reduce the residuals to the level of round-off error in the computer. However, as N increases the effectiveness of the scheme deteriorates, and the method may only converge in the asymptotic sense that at first the errors become smaller, but later they increase without limit. The numerical data establish that the most reliable results are calculated for degrees in the range $20 \leq N \leq 32$. Since in many cases the residuals decrease indefinitely, the system of discrete equations implemented in the NSTAB code does seem in general to have a solution. It is more difficult to interpret numerical results for methods in which that is not the case (9).

Comparison with Experiment

In addition to convergence studies, an important test of numerical methods is comparison with experiment. Recent communications from Japan about the LHD stellarator there state that average values of β as high as 3.2% have been observed (10). The success of the LHD experiment in achieving such good performance shows that stellarators have desirable physical properties despite the nonexistence of smooth solutions of the system of partial differential equations for MHD equilibrium. Comparison with the measurements increases confidence in the mathematical model of a weak solution approximated asymptotically by numerical computations.

NSTAB calculations modeling the LHD experiment were performed with the magnetic axis assumed to have major radius $R = 3.6$ m and with the radial mesh size equal to 1/27. For a triangular pressure profile $p = p_0(1 - s^{1/2})$ like measured distributions of the electron temperature, the numerical results agreed with the observations, but in some respects the LHD

Table 3. Relationship between the highest degree N of the spectral terms used in runs of NSTAB, the number of iterations that were performed successfully for this value of N, and the amount that the residuals decayed

N	16	20	24	32	48
Cycle	1×10^6	5×10^5	1×10^5	5×10^4	1×10^4
Error	10^{-9}	10^{-7}	10^{-5}	10^{-4}	10^{-3}

In the finite difference scheme 14 intervals of the radial coordinate s were used.

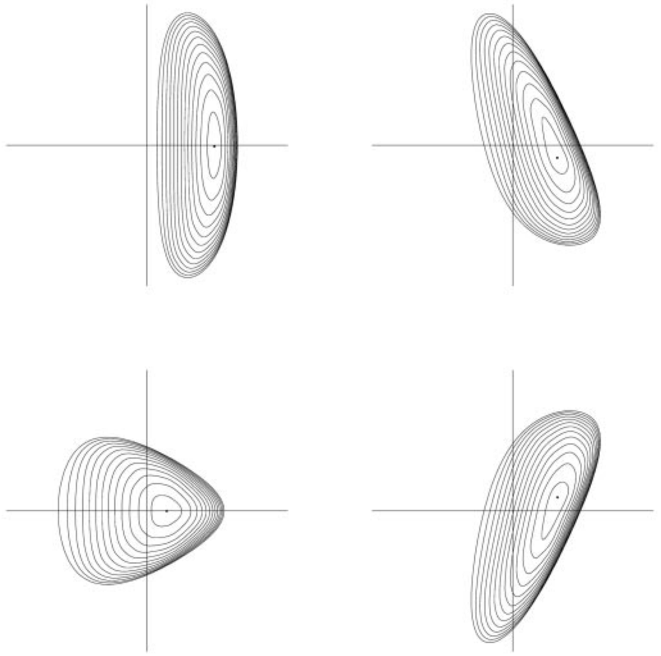

Fig. 2. Four cross sections of the flux surfaces over the full torus of a quasiaxially symmetric version of the W7-AS stellarator at $\beta = 0.04$ with net current bringing the rotational transform into the interval $0.65 > \iota > 0.30$. In this example the NSTAB code was run 5×10^4 cycles with spectral terms of degrees 24 and 40 in the poloidal and toroidal angles.

outperformed theoretical predictions. Perhaps it makes sense to think of the computer codes not as self-consistent physics, but as a mathematically correct simulation to be adjusted like a scaling law to fit laboratory data. Runs of the NSTAB code over several field periods of the LHD establish that modes of low order become linearly unstable for moderate choices of β, but they remain nonlinearly stable so that the plasma stays contained within an outer pedestal of good confinement and can access a second stability regime.

Measurements from the Wendelstein 7-AS (W7-AS) stellarator experiment at the Max Planck Institute for Plasma Physics in Garching, Germany are reported to have reached average values of β near 3% after improvements were made in the divertor. In Fig. 2 we display our calculation of a quasiaxially symmetric equilibrium found by making minor changes, given by Table 4, in coefficients Δ_{mn} defining the separatrix of the W7-AS. The NSTAB code predicts good nonlinear stability of these configurations, and the results are consistent with the observations. Moreover, experimental data at smaller values of β validate an algorithm implemented in the NSTAB code to estimate the bootstrap current (11, 12).

In designing quasisymmetric configurations like the MHH2 it is helpful to recognize that pessimistic forecasts about stability of the W7-AS have not been borne out. It turns out that the β limit of the W7-AS is remarkably high, and for these stellarators equilibrium seems to be more of a problem than stability. Similarities in the physics of the W7-AS and the MHH2 suggest that successes of the W7-AS experiment may be reflected in

Table 4. Fourier coefficients Δ_{mn} of a quasiaxially symmetric modification of the W7-AS defined by the formula $r + iz = e^{iu} \sum \Delta_{mn}\, e^{-imu+5inv}$

$m \backslash n$	−2	−1	0	1	2	3
−1	0.000	0.000	0.000	0.015	0.000	0.000
0	0.000	0.000	1.000	0.065	−0.020	0.000
1	−0.015	0.110	12.200	0.250	0.010	−0.005
2	0.000	0.000	−0.300	−0.310	0.060	0.000
3	0.000	0.010	0.045	−0.045	0.025	0.000
4	0.000	0.000	−0.010	−0.010	0.005	0.000

good performance of the MHH2. We conclude that stability is not such a serious issue, so more attention can be devoted to designing smooth coils tailored to provide robust magnetic surfaces in a new experiment with an effective divertor.

Ballooning modes have been observed in the Tokamak Fusion Test Reactor at the Princeton Plasma Physics Laboratory (Princeton, NJ) for shots at $\beta = 0.007$ with peaky pressure profiles (13). To simulate those measurements we have made runs of the NSTAB code for a circular torus of aspect ratio three, subdivided into eight fictitious field periods, with $p = p_0(1 - s)^5$ and $1.04 > \iota > 0.40$. A bifurcated equilibrium was calculated over one such field period that had ripple in the flux surfaces produced by using trigonometric polynomials of poloidal and toroidal degrees $m = 7$ and $n = 8$ to trigger a radially localized MHD mode. The NSTAB iterative scheme converged so well that the residuals decayed to the level 10^{-13} of round-off error, which established the existence of a solution without axial symmetry of the discrete problem implemented in the code. This computation substantiates the ability of the mountain pass theorem to predict MHD activity in magnetic fusion experiments, but the convergence to bifurcated solutions is usually slow.

Calculations of nonlinear stability using the NSTAB code agree well with experimental observations in both stellarators and tokamaks. The same is not always true of local theories such as the ballooning criterion evaluated by solving an ordinary differential equation along magnetic lines (14, 15). We have performed numerical computations showing that that method gives results for the β limit in the LHD significantly lower than values that have been observed. Our convergence studies demonstrate that in stellarators a more realistic prediction of β limits for ideal MHD modes of high order is obtained from runs of the NSTAB code that determine bifurcated equilibria whose magnetic surfaces have a ballooning structure. The techniques we have developed to perform these calculations effectively on fine grids using spectral terms of high degree have allowed us to optimize the design of the MHH2 configuration so that it has desirable physical properties combined with three-dimensional geometry that is not too complicated. Induced current can be used to operate the device successfully as a stellarator-tokamak hybrid, and we have found that a β limit of 5% prevails if the rotational transform is not allowed to rise much above 1/2.

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